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On the possible values of the orbit distance between a near-Earth asteroid and the Earth

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ABSTRACT

We consider all the possible trajectories of a near-Earth asteroid (NEA), corresponding to the whole set of heliocentric orbital elements with perihelion distance $q \le 1.3$ au and eccentricity $e \le 1$ (NEA class). For these hypothetical trajectories, we study the range of the values of the distance from the trajectory of the Earth (assumed on a circular orbit) as a function of selected orbital elements of the asteroid. The results of this geometric approach are useful to explain some aspects of the orbital distribution of the known NEAs. We also show that the maximal orbit distance between an object in the NEA class and the Earth is attained by a parabolic orbit, with apsidal line orthogonal to the ecliptic plane. It turns out that the threshold value of q for the NEA class ($q_{\max} = 1.3$ au) is very close to a critical value, below which the above result is not valid.

'Nothing was visible, nor could be visible, to us, except Straight Lines', E. A. Abbott, *Flatland*.

Key words: surveys – celestial mechanics – minor planets, asteroids: general.

1 INTRODUCTION

In this paper, we consider all the possible trajectories of a near-Earth asteroid (NEA), that is the whole set of trajectories with perihelion distance $q \le 1.3$ au and eccentricity $e \le 1$. We study the range of the values of the orbit distance between hypothetical asteroids on these trajectories and the Earth, which is assumed on a circular orbit.

The orbit distance between an asteroid and the Earth, here denoted by $d_{\min}$, is the minimum value of the distance between two points on the two orbits; it is also called MOID[1] in the literature. This distance plays an important role to understand whether an asteroid can impact the Earth; for this purpose the category of potentially hazardous asteroids has been introduced (see Bowell & Muinonen 1994), which are NEAs with $d_{\min} \le 0.05$ au and absolute magnitude $H \le 22$. The value of H is related to the size of the asteroid and the constraint $H \le 22$ is set to take into account objects which are large enough to produce serious damages in the case of impact on our planet.

The conversion between the size and absolute magnitude of an asteroid strongly depends on the albedo (measuring the reflectivity properties of the body), whose value is badly known for almost all asteroids. For this reason, we prefer to speak of *bright* (H small) and *faint* asteroids (H large).

The orbit distance $d_{\min}$ is also important to understand whether a faint asteroid can be observed from the Earth; hereafter we deal with this issue. If $d_{\min}$ is large, then the asteroid is hard to detect. In contrast, if $d_{\min}$ is small, in most cases the asteroid will get close enough to the Earth and will be detected, provided we wait long enough.

Of course, other factors intervene in an asteroid detection: weather, solar elongation of the observed body, etc.; however, we shall show that a purely geometric argument is enough to explain some selection effects in the orbital distribution of the observed population of NEAs.

In Fig. 1, we plot the values of the perihelion distance and the perihelion argument, that is the pairs (q, ω), for all the 9220 known NEAs in the NEODyS database (http://newton.dm.unipi.it/neodys) to the date of 2012 October 21. The grey dots represent asteroids with $H \le 22$, while the black dots are those with $H > 22$.

We immediately observe that most of the fainter asteroids are grouped together in a peculiar way.

In Fig. 2, we plot the values of the pairs (q, $d_{\min}$); also here the grey dots are asteroids with $H \le 22$ and the black ones with $H > 22$. In this case a V-shaped structure appears, composed of two lines: the fainter NEAs accumulate towards the line with $q > 1$ au, while that with $q < 1$ au is mostly due to the brighter NEAs.

In this paper, we shall prove some facts concerning the distance between two confocal conics, and we shall see that these purely geometric results can give an explanation of the features appearing in Figs 1 and 2.

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[1] Minimum orbit intersection distance.

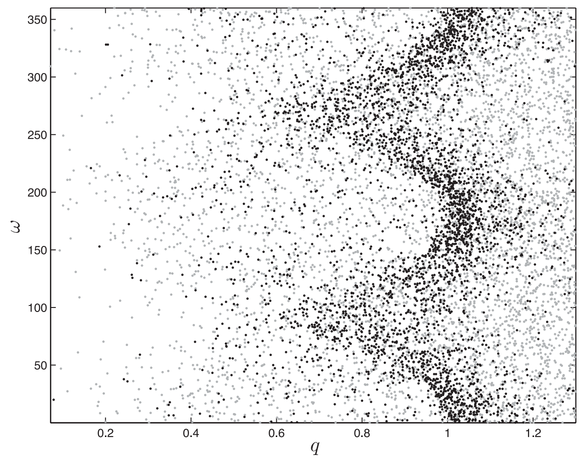

Figure 1. Orbital distribution of the known NEAs in the plane (q, ω). The black dots correspond to the fainter asteroids, which have absolute magnitude $H > 22$. The grey dots represent all the others.

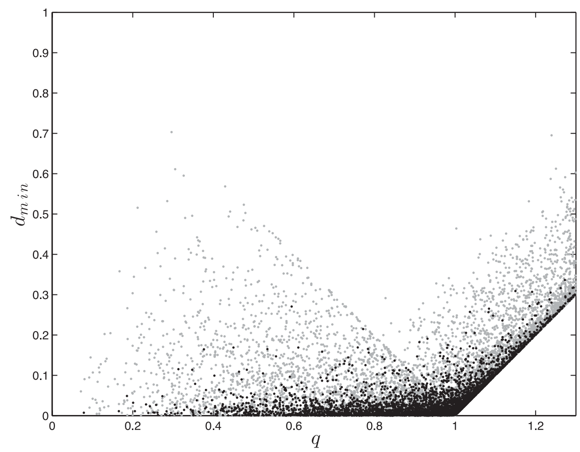

Figure 2. Orbital distribution of the known NEAs in the plane (q, $d_{\rm min}$). The black dots correspond to the fainter asteroids ($H > 22$).

The structure of the paper is as follows. In Section 2, we introduce preliminary facts and notations. In Section 3, we prove the geometric results. Section 4 is devoted to using these results and explaining the orbital distribution of the known NEAs. In Section 5, we discuss the perihelion distance threshold of the NEA class, from the point of view of the orbit distance. In Section 6, we give a formula for the apparent magnitude of a hypothetic asteroid whose orbit attains the maximal orbit distance in the NEA class, and is observed at its closest point to the Earth. Finally, in Appendix A, we give some complementary results and computations.

2 DEFINITIONS

2.1 Preliminaries

We denote by $\mathcal{N}$ the set of possible trajectories of NEAs; these can be described by cometary orbital elements $E = (q, e, I, \Omega, \omega)$ with the constraints

$$0 \leq q \leq q_{\rm max} = 1.3 \text{ au}, \qquad 0 \leq e \leq 1. \tag{1}$$

Here q is the perihelion distance, e the eccentricity, I the inclination, Ω the longitude of the ascending node and ω the argument of perihelion. We adopt this set of elements because they naturally lend themselves to the description of parabolic orbits, which are included in the set $\mathcal{N}$.

On the other hand, if $q = 0$, the trajectory is either pointwise or rectilinear, and using cometary orbital elements we can neither distinguish these two cases, nor characterize two rectilinear trajectories with different lengths. Moreover, if $I = 0, \pi$ or $e = 0$ the angle ω is not defined.

If $I = 0$ also Ω is not defined, but this orbital element will not play a role in the following.

We assume that the Earth moves on a circular orbit, whose elements $E' = (q', e', I', \Omega', \omega')$ are set as follows:

$$q' = 1 \text{ au}, \qquad e' = I' = \Omega' = \omega' = 0.$$

Hereafter we will denote by $\mathcal{A}$ the trajectory of an asteroid in the NEA class, and by $\mathcal{A}'$ that of the Earth.

Moreover, for $q \neq 0$ and $I \neq 0, \pi$, we introduce the ascending/descending nodal distances

$$d^{\pm}_{\rm nod} = q' - \frac{q(1+e)}{1 \pm e\cos\omega}. \tag{2}$$

2.2 The orbit distance ($d_{\rm min}$)

Let us consider two celestial bodies on confocal Keplerian orbits. We fix a reference frame, with origin in the common focus and let (E, v), (E', v') be the sets of orbital elements of the bodies. Here E, E' describe the trajectories of the orbits and v, v' are parameters along them, e.g. the true anomalies f, f'. We denote by $\mathcal{E} = (E, E')$ the two-orbit configuration and by $V = (v, v')$ the vector of the orbit parameters. Moreover, we write $\mathcal{X} = \mathcal{X}(E, v)$, $\mathcal{X}' = \mathcal{X}'(E', v')$ for the Cartesian coordinates of the two bodies.

For a given two-orbit configuration $\mathcal{E}$, we introduce the *Keplerian distance function* d, defined by

$$V \mapsto d(\mathcal{E}, V) = |\mathcal{X} - \mathcal{X}'|, \tag{3}$$

where $V \in \mathbb{T}^2$ (the two-dimensional torus) if $e < 1$, $V \in (-\pi, \pi) \times S^1$ if $e = 1$. Here $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$.

The local minimum points of d can be found by computing all the stationary points of d^2, as in Gronchi (2005) or Baluyev & Kholshevnikov (2005), where the authors use algebraic tools such as resultants and Gröbner's bases. See also Gronchi (2002), Kholshevnikov & Vassiliev (1999).

Apart from the case of two concentric coplanar circles or two overlapping ellipses, the function d^2 has finitely many stationary points Gronchi (2005). We can find configurations with up to four local minima of d^2; this is thought to be the maximum possible.

Let $V_h = V_h(\mathcal{E})$ be a local minimum point of $V \mapsto d^2(\mathcal{E}, V)$. Then, following Gronchi & Tommei (2007), we consider the maps

$$\begin{aligned} &\mathcal{E} \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) && \text{(local minimal distance)}, \\ &\mathcal{E} \mapsto d_{\rm min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) && \text{(orbit distance)}. \end{aligned} \tag{4}$$

For each choice of the configuration $\mathcal{E}$, $d_{\rm min}(\mathcal{E})$ gives the orbit distance.

The maps d_h, $d_{\rm min}$ are not differentiable where they vanish. This singularity has been studied in Gronchi & Tommei (2007) to define a meaningful uncertainty of d_h, $d_{\rm min}$, with possibly negative values of these maps. However, this will not constitute a problem in this work.

The maps d_h may have other singularities due to bifurcations. Moreover, $d_{\rm min}$ can lose regularity when two local minima exchange their role as absolute minimum. A detailed analysis of these phenomena is still lacking. The occurrence of bifurcations will not be

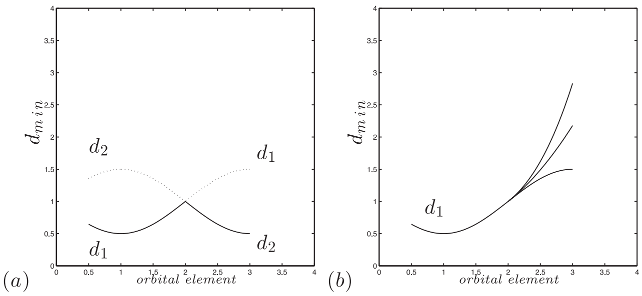

Figure 3. Left: exchange of local minimum points as absolute minimum. Right: bifurcation of a minimum point.

a problem as well, since at bifurcation points the derivative of d_h with respect to an orbital element can be extended with regularity, see Fig. 3(b). In contrast, possible exchanges of local minimum points as absolute minimum make things more difficult because the monotonicity properties of the maps d_h may not hold for $d_{\min}$, see Fig. 3(a). We organize the proofs in the paper so that there is no need to know if and where such a singularity occurs.

3 GEOMETRIC RESULTS

The maximum of the orbit distance $d_{\min}$ in the NEA class $\mathcal{N}$ does exist. Below we identify the NEA orbits attaining this maximum and completely characterize the possible values of $d_{\min}$ assuming the values of some orbital elements.

Since $\mathcal{A}'$ is circular we can set $\Omega = 0$ in computing the orbit distance between $\mathcal{A}$ and $\mathcal{A}'$. Thus, the elements of $\mathcal{E}$ that can vary are only q, e, I, ω. Moreover, taking advantage of the symmetry, we can restrict our analysis to $\omega \in [0, \pi/2]$, $I \in [0, \pi/2]$.

3.1 The maximal orbit distance

If $\mathcal{A}$ is either pointwise or rectilinear, then the maximal orbit distance is $q' = 1$ au. Assume $q \neq 0$. We parametrize the trajectories $\mathcal{A}$, $\mathcal{A}'$ as follows:[2]

$$x = r\cos(f+\omega), \qquad y = r\sin(f+\omega)\cos I,$$

$$z = r\sin(f+\omega)\sin I,$$

with

$$r = \frac{q(1+e)}{1+e\cos f},$$

and

$$x' = q'\cos f', \qquad y' = q'\sin f', \qquad z' = 0.$$

This choice of coordinates implies that the mutual nodal line coincides with the x-axis. We denote the position vector of the asteroid by

$$\boldsymbol{r} = r\hat{\boldsymbol{r}},$$

where

$$\hat{\boldsymbol{r}} = (\cos(f+\omega), \sin(f+\omega)\cos I, \sin(f+\omega)\sin I).$$

[2] With this parametrization, we obtain only pointwise trajectories for $q = 0$. If $e = 0$ or $I = 0$ every choice of ω produces the same trajectory.

The squared distance function is

$$\begin{aligned} d^2(\mathcal{E}, V) &= (x-x')^2 + (y-y')^2 + z^2 \\ &= \frac{q^2(1+e)^2}{(1+e\cos f)^2} + q'^2 - \\ &\quad -\frac{2qq'(1+e)}{1+e\cos f}[\cos f'\cos(f+\omega) + \sin f'\sin(f+\omega)\cos I]. \end{aligned}$$

The orbit distance $d_{\min}$ is the value of d at one of the solutions of

$$\frac{\partial d^2}{\partial f} = 0, \qquad \frac{\partial d^2}{\partial f'} = 0, \tag{5}$$

with[3]

$$\begin{aligned} \frac{\partial d^2}{\partial f} &= 2(x-x')\frac{\partial x}{\partial f} + 2(y-y')\frac{\partial y}{\partial f} + 2z\frac{\partial z}{\partial f} \\ &= 2\left[\frac{z}{\sin I}(x' - x\cos^2 I) + x(y-y')\cos I\right] \\ &\quad + 2\frac{e\sin f}{1+e\cos f}(x^2+y^2+z^2-xx'-yy') \end{aligned}$$

$$\frac{\partial d^2}{\partial f'} = -2(x-x')\frac{\partial x'}{\partial f'} - 2(y-y')\frac{\partial y'}{\partial f'} = 2(xy' - x'y),$$

where we have used the relations

$$\frac{\partial x}{\partial f} = -\frac{z}{\sin I} + xg(f,e),$$

$$\frac{\partial y}{\partial f} = x\cos I + yg(f,e),$$

$$\frac{\partial z}{\partial f} = x\sin I + zg(f,e),$$

with

$$g(f,e) = \frac{e\sin f}{1+e\cos f}, \tag{6}$$

and

$$\frac{\partial x'}{\partial f'} = -y', \qquad \frac{\partial y'}{\partial f'} = x'.$$

Now we prove the following result.

Proposition 1. We have

$$\max_{\mathcal{N}} d_{\min} = D(\bar{x}),$$

where

$$D(x) = \sqrt{(x-q')^2 + \left(\frac{x^2 - 4q_{\max}^2}{4q_{\max}}\right)^2} \tag{7}$$

and $\bar{x}$ is the (unique) real solution of

$$x^3 + 4q_{\max}^2 x - 8q'q_{\max}^2 = 0. \tag{8}$$

We have $\bar{x} \approx 1.5$ au and $D(\bar{x}) \approx 1.0011$ au.

Proof: we easily find that if $\mathcal{A}$ is a pointwise or a rectilinear trajectory then the value of the orbit distance is at most $q' = 1$au.

Now we consider trajectories with $q \neq 0$ (neither rectilinear nor pointwise). Denote by $\mathcal{A}_*$ a trajectory in $\mathcal{N}$ attaining the maximal orbit distance. We divide the proof into five steps.

[3] Since $\frac{z}{\sin I} = \frac{p\sin(f+\omega)}{1+e\cos f}$, we can consider the expression $\frac{z}{\sin I}$ meaningful also for $I = 0$.

Step 1. We can choose $I = \pi/2$.

We compute the derivative of a squared local minimal distance:

$$\frac{\partial d_h^2}{\partial I}(\mathcal{E}) = \frac{\partial d^2}{\partial I}(\mathcal{E}, V_h) + \frac{\partial d^2}{\partial V}(\mathcal{E}, V_h)\frac{\partial V_h}{\partial I}(\mathcal{E}). \tag{9}$$

Since $V_h = (f_h, f_h')$ is a stationary point of $V \mapsto d^2(\mathcal{E}, V)$, (9) reduces to

$$\frac{\partial d_h^2}{\partial I}(\mathcal{E}) = \frac{\partial d^2}{\partial I}(\mathcal{E}, V_h) = 2y'(f_h')z(f_h)\,, \tag{10}$$

which has the same sign as $\sin f_h' \sin(f_h + \omega)$. Therefore, if $d_h = d_{\min}$, then d_h is a non-decreasing function of I for $0 < I < \pi/2$; in fact for $0 < I < \pi/2$ the minimal distance can be attained only if $y'(f'_{\min})z(f_{\min}) \geq 0$ because we have

$$d_h^2 = q'^2 + r^2(f_h) - 2q'r(f_h)[\cos(f_h + \omega)\cos f_h' + \sin(f_h + \omega)\sin f_h' \cos I]$$

and, if $\sin(f_h + \omega)\sin f_h' < 0$, then by changing f_h' with $-f_h'$ we would obtain a value of d smaller than d_h. This implies that the maximal value of d_h is attained for $I = \pi/2$. Note that, even if there were an exchange of local minimum points as absolute minimum when I varies, then from the computations above this would not matter; in fact, all the local minimal distances have the same monotonicity properties as functions of I.

Step 2. We must have $y'(f'_{\min}) = 0$. Moreover, if $e \neq 0$, then $\mathcal{A}_*$ must have either $\omega = 0$ or $\omega = \pi/2$.

First, we observe that the second equation in (5) for $I = \pi/2$ gives $x(f_h)y'(f_h') = 0$. We exclude the solutions with $x(f_h) = 0$, which corresponds to two points $P_\pm$ on the z-axis, since they cannot attain the minimal distance $d_{\min}$. In fact, we observe that if $x(f_h) = 0$ then $d_h = \sqrt{z_h^2 + q'^2}$ with $z_h = z(f_h) \geq q$, whatever the value of f_h'. On the other hand, since $q \neq 0$,

$$d_{\min} \leq \min|d^\pm_{\rm nod}| < \min\sqrt{q'^2 + q^2\left(\frac{1+e}{1 \pm e\cos\omega}\right)^2} \leq \sqrt{q'^2 + q^2} \leq \sqrt{q'^2 + z_h^2} = d_h.$$

Thus, we must have $y'(f'_{\min}) = 0$ and, if $e \neq 0$, we have to choose ω that maximizes the minimum of the two distances between the nodal points $Q_\pm \equiv (\pm q', 0, 0)$ and $\mathcal{A}$. We select one nodal point, say Q_+; by symmetry, the discussion for the distance between $\mathcal{A}$ and Q_- can be included by extending to $\omega \in [0, \pi]$ the study of the distance between $\mathcal{A}$ and Q_+.

This distance is given by minimizing the function

$$f \mapsto d^2_{Q_+}(f, e, \omega) = r^2 - 2rq'\cos(f + \omega) + q'^2\,,$$

with

$$r = r(f, e) = \frac{q(1+e)}{1 + e\cos f}$$

and $f \in S^1$ if $e < 1$, $f \in (-\pi, \pi)$ if $e = 1$. The stationarity condition for $f \mapsto d^2_{Q_+}(f, e, \omega)$ gives

$$\frac{\partial d^2_{Q_+}}{\partial f} = 2\left(r\frac{\partial r}{\partial f} - \frac{\partial r}{\partial f}q'\cos(f + \omega) + rq'\sin(f + \omega)\right) = 0\,, \tag{11}$$

with

$$\frac{\partial r}{\partial f} = rg\,, \tag{12}$$

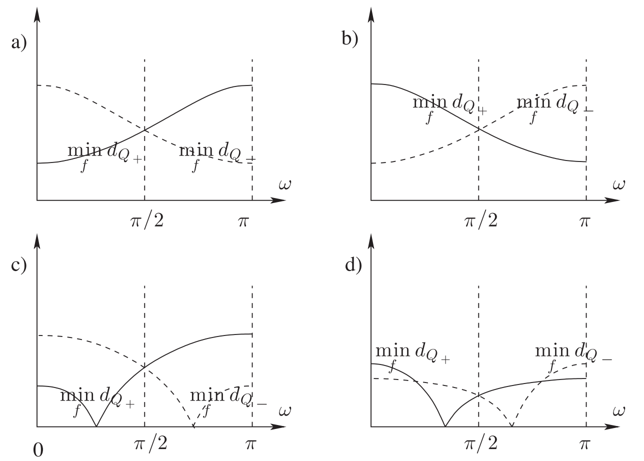

Figure 4. Some possible graphs of the distances between $Q_\pm$ and $\mathcal{A}$, denoted by $\min_f d_{Q_\pm}$, as a function of ω. The plot of $\min_f d_{Q_-}$ is dashed.

and g defined as in (6). Let f_h be a minimum point of $f \mapsto d^2_{Q_+}(f, e, \omega)$; then we write

$$d^2_{Q_+,h}(e, \omega) = d^2_{Q_+}(f_h(e, \omega), e, \omega)\,,$$

which gives the local minimal distance from Q_+ assuming, as we did, $I = \pi/2$. The derivative

$$\frac{\partial d^2_{Q_+,h}}{\partial\omega}(e, \omega) = \frac{\partial d^2_{Q_+}}{\partial\omega}(f_h(e, \omega), e, \omega) = 2r_h q'\sin(f_h + \omega)\,, \tag{13}$$

with $r_h = r(f_h, e)$, can vanish only if $\sin(\omega + f_h) = 0$. By using (11) and (12) we obtain that

$$\frac{\partial d^2_{Q_+,h}}{\partial\omega}(e, \omega) = 0 \quad \text{if and only if} \quad g_h(r_h \mp q') = 0\,,$$

with $g_h = g(f_h, e)$. Thus, the derivative (13) can vanish only for $g(f_h, e) = 0$, which together with $\sin(f_h + \omega) = 0$ gives $\sin\omega = 0$ (since $e \neq 0$), or for $r(f_h, e) = q'$, which implies $d_{Q_+,h} = 0$.

By a classical result on the distance between a point and a conic section in the plane (see Section A1 for details), for each $\omega \in [0, \pi]$ the only exchange of local minimum points $f_h(e, \omega)$ as absolute minimum of d_{Q_+} can occur for $\omega = 0, \pi$.

Let us fix a value for $e \neq 0$. From the above discussion it follows that there are three possible cases (see Fig. 4): (1) $\omega \mapsto \min_f d_{Q_+}(e, \omega)$ is increasing over $(0, \pi)$; (2) $\omega \mapsto \min_f d_{Q_+}(e, \omega)$ is decreasing over $(0, \pi)$; (3) there exists $\bar\omega \in (0, \pi)$ such that $\omega \mapsto \min_f d_{Q_+}(e, \omega)$ is decreasing over $(0, \bar\omega)$, vanishes at $\omega = \bar\omega$ and is increasing over $(\bar\omega, \pi)$.

Moreover, by symmetry we have

$$\min_f d_{Q_-}(e, \omega) = \min_f d_{Q_+}(e, \pi - \omega).$$

In Fig. 4, we show some of the possible graphs of $\min_f d_{Q_+}$ and $\min_f d_{Q_-}$. We recall that

$$d_{\min} = \min\{\min_f d_{Q_+}, \min_f d_{Q_-}\}\,;$$

therefore, we conclude that, for $e \neq 0$, the map

$$[0, \pi/2] \ni \omega \mapsto d_{\min}(e, \omega)$$

attains its maximum value either for $\omega = 0$ or for $\omega = \pi/2$.

Step 3. $\mathcal{A}_*$ must have all the nodes external to $\mathcal{A}'$; hence, $\omega = \pi/2$.

If not, we have $d_{\min} \leq 1$ au and this cannot be a configuration which maximizes $d_{\min}$ in $\mathcal{N}$. In fact the parabolic trajectory $\mathcal{P}$, with

$I = \pi/2$, $\omega = \pi/2$, $q = q_{\rm max}$, gives the larger value $d_{\rm min} = D(\bar{x}) \approx 1.0011$ au, defined by (7), (8); the equation of the trajectory of $\mathcal{P}$ in the plane (x, z) can be written as

$$z = \frac{x^2 - 4q_{\rm max}^2}{4q_{\rm max}} ;$$

thus, the distance of the points of the parabola from $(x, z) \equiv (q', 0)$ is given by (7). The stationary points equation for the function $D(x)$ defined in (7) is (8), and it has only one solution $\bar{x} > 0$, which corresponds necessarily to the absolute minimum point.

Since the nodes of $\mathcal{A}_*$ are external to $\mathcal{A}'$, we exclude the value $\omega = 0$, which gives $d_{\rm min} = q - q'$, which is smaller than the orbit distance for $\omega = \pi/2$.

Step 4. $\mathcal{A}_*$ must have $q = q_{\rm max}$.

By Step 2, we can consider in the plane (x, z) the problem of minimizing the distance between a point in $\mathcal{A}$ and a nodal point $Q_\pm$. If we change q into $q + \delta q$, with $\delta q > 0$, the modified trajectory encloses the original one. By Step 3, the nodes of $\mathcal{A}_*$ are external to $\mathcal{A}'$; hence, every line joining either Q_+ or Q_- to a point of the modified trajectory must cross the original trajectory, and we conclude that both distances of $Q_\pm$ from $\mathcal{A}$ increase with q. This implies that the maximal value of every d_h, and hence also of $d_{\rm min}$, is attained for $q = q_{\rm max}$.

Step 5. $\mathcal{A}_*$ must have $e = 1$.

We have

$$\frac{\partial \boldsymbol{r}}{\partial e} = \frac{\partial r}{\partial e}\hat{\boldsymbol{r}}\,, \qquad \frac{\partial r}{\partial e} = \frac{q(1-\cos f)}{(1+e\cos f)^2} \geq 0. \tag{14}$$

Moreover, by Step 3, the nodes of $\mathcal{A}_*$ are external to $\mathcal{A}'$. Using $I = \pi/2$ (which holds by Step 1) and $y'(f'_{\rm min}) = 0$ (by Step 2), we conclude that the maximum value of $d_{\rm min}$ is attained at $e = 1$.

3.2 Optimal bounds for $d_{\rm min}$

We study the range of the possible values of $d_{\rm min}$ by assuming the values of selected orbital elements.

First we establish the possible values of $d_{\rm min}$ as a function of (q, ω).

Proposition 2. Let $\mathcal{D}_1 = \{(e, I) : 0 \leq e \leq 1, 0 \leq I \leq \frac{\pi}{2}\}$, $\mathcal{D}_2 = \{(q, \omega) : 0 < q \leq q_{\rm max}, 0 \leq \omega \leq \frac{\pi}{2}\}$. For each choice of $(q, \omega) \in \mathcal{D}_2$, we have

$$\begin{cases} \min\limits_{(e,I)\in\mathcal{D}_1} d_{\rm min} = \max\{0, q - q'\} \\ \max\limits_{(e,I)\in\mathcal{D}_1} d_{\rm min} = \max\{q' - q, \delta_\omega(q, \omega)\} \end{cases}, \tag{15}$$

where $\delta_\omega(q, \omega)$ is the distance between $\mathcal{A}'$ and $\mathcal{A}$ with $e = 1$, $I = \pi/2$;

$$\delta_\omega(q, \omega) = \sqrt{(\xi - q'\sin\omega)^2 + \left(\frac{\xi^2 - 4q^2}{4q} + q'\cos\omega\right)^2}, \tag{16}$$

with $\xi = \xi(q, \omega)$ the unique real solution of

$$x^3 + 4q(q + \cos\omega)x - 8q'q^2\sin\omega = 0.$$

Proof. Lower bound: by Step 1 in Proposition 1, we can choose $I = 0$. If $q \leq q'$, for suitable choices of e we have $d_{\rm min} = 0$. If $q > q'$, then $d_{\rm min}$ is always greater than or equal to $q - q'$ and it is equal to $q - q'$ for $e = 0$.

Upper bound: by Step 1 in Proposition 1, given (q, ω), we can choose $I = \pi/2$. Thus, we have to properly choose only the value of e. by Step 2 in Proposition 1, we have $y'(f'_{\rm min}) = 0$. Moreover, the relation

$$\frac{\partial}{\partial e} d^\pm_{\rm nod} = -\frac{q(1 \mp \cos\omega)}{(1 \pm e\cos\omega)^2} \leq 0$$

holds so that, given (q, ω), if there exists $\bar{e} \in [0, 1)$ such that the nodes of $\mathcal{A}$ with $e = \bar{e}$ are external to $\mathcal{A}'$, then the nodes remain external for each $\mathcal{A}$ with $e \in [\bar{e}, 1]$. Therefore, for each choice of (q, ω), the maximal value of $d_{\rm min}$, for e such that the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$, is attained for $e = 1$. This can be proven as in Step 5 of Proposition 1. On the other hand, the maximal value of $d_{\rm min}$, for e such that at least one node of $\mathcal{A}$ is internal to $\mathcal{A}'$, is attained for $e = 0$. In fact, in the case of an internal node we necessarily have $q < q'$, so that the maximal value of $d_{\rm min}$ is $q' - q$. In fact we have

$$d_{\rm min} \leq \min |d^\pm_{\rm nod}| \leq q' - q$$

and $d_{\rm min} = q' - q$ for $e = 0$.

Remark 1. We note that, if the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$, then necessarily $q > q'/2$. In fact we can compute

$$\min\{q \in [0, q_{\rm max}] : \exists (e, \omega) \in [0, 1] \times [0, 2\pi] : \max d^\pm_{\rm nod} \leq 0\}$$

from the relations

$$\begin{aligned} 0 &\geq \min_{e\in[0,1]} \min_{\omega\in[0,2\pi]} \max\left(q' - \frac{q(1+e)}{1 \pm e\cos\omega}\right) \\ &= \min_{e\in[0,1]} \min_{\omega\in[0,\pi/2]} \left(q' - \frac{q(1+e)}{1 + e\cos\omega}\right) \\ &= \min_{e\in[0,1]} (q' - q(1+e)) = q' - 2q. \end{aligned}$$

In Fig. 5, we plot the graph of the function $(q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\rm min}(q, \omega)$, and in Fig. 6, we show some level curves of $\max_{\mathcal{D}_1} d_{\rm min}$. In the latter, we also draw the curve γ which separates the region, in the plane (q, ω), where the orbits maximizing $d_{\rm min}$ have $e = 0$, from the region where such orbits have $e = 1$. To explicitly compute γ, consider the system

$$\begin{cases} (q' - q)^2 = (x - q'\sin\omega)^2 + \left(\frac{(x^2 - 4q^2)}{4q} + q'\cos\omega\right)^2 \\ x^3 + 4q(q + \cos\omega)x - 8q^2q'\sin\omega = 0 \end{cases}. \tag{17}$$

We use resultant theory, see Cox, Little & O'Shea (1992), to eliminate the variable x and obtain the algebraic curve,

$$\begin{aligned} &2q^4 + 2q'(-5 + 7y)q^3 - 2q'^2(3y + 22)(y - 1)q^2 \\ &+ q'^3(y^3 + 13y^2 + 9y - 27)q - 2q'^4y^3 = 0, \end{aligned} \tag{18}$$

with $y = \cos\omega$. Additional details on this computation are given in Section A2.

Now we introduce a partition of $\{(q, e) \in (0, q_{\rm max}] \times [0, 1]\}$ in three regions, an inner region (IR), a crossing region (CR) and an outer region (OR):

$$\begin{aligned} \text{IR} &= \{(q, e) : Q < q'\}, \\ \text{CR} &= \{(q, e) : Q \geq q', q \leq q'\}, \\ \text{OR} &= \{(q, e) : q > q'\}, \end{aligned}$$

where $Q = q(1 + e)/(1 - e)$ is the aphelion distance (see Fig. 8). In regions IR, OR the trajectory $\mathcal{A}$ is totally inside, respectively outside, the ball $\mathcal{B}(0, q')$ with centre O and radius q'. In region CR, crossings of $\mathcal{A}$, $\mathcal{A}'$ are possible. Region CR can be divided into two sub-regions:

$$\text{CR}_1 = \{(q, e) : Q \geq q', q \leq q', q(1 + e) \leq q'\},$$

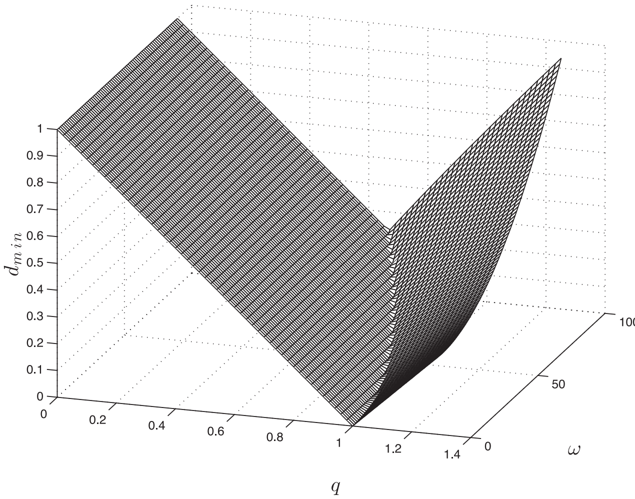

Figure 5. Graph of the function $(q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\min}(q, \omega)$.

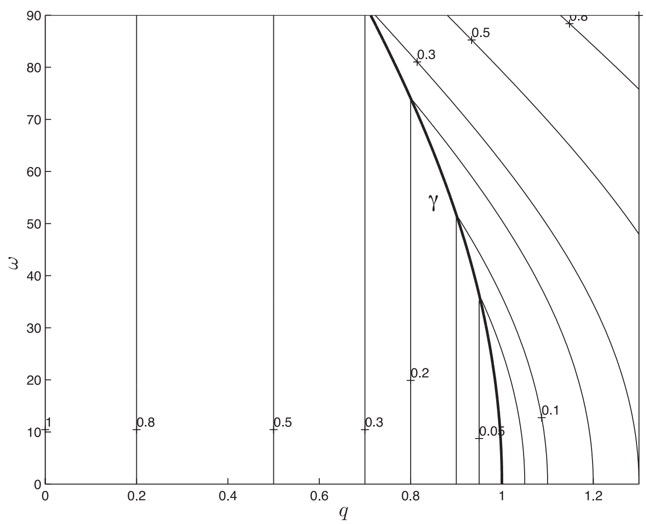

Figure 6. Level curves of $(q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\min}(q, \omega)$. We also plot the curve γ (enhanced in the figure) defined in (18).

$$\mathrm{CR}_2 = \{(q, e) : Q \geq q', q \leq q', q(1+e) > q'\}.$$

According to the value of I, ω, with $I \neq 0$, in region CR_1 we have either two internal nodes or only one internal and in region CR_2 we have either two external nodes or only one internal.

Now we describe the possible values of $d_{\min}$ as a function of (q, e).

Proposition 3. Let $\mathcal{D}_3 = \{(I, \omega) : 0 \leq I \leq \frac{\pi}{2}, 0 \leq \omega \leq \frac{\pi}{2}\}$, $\mathcal{D}_4 = \{(q, e) : 0 < q \leq q_{\max}, 0 \leq e \leq 1\}$. For each choice of $(q, e) \in \mathcal{D}_4$ we have

$$\begin{cases} \min\limits_{(I,\omega)\in\mathcal{D}_3} d_{\min} = \max\{0, q' - Q, q - q'\} \\ \max\limits_{(I,\omega)\in\mathcal{D}_3} d_{\min} = \max\{\min\{q' - q, Q - q'\}, \delta_e(q, e)\} \end{cases}, \tag{19}$$

where $Q = q(1+e)/(1-e)$ is the aphelion distance and $\delta_e(q, e)$ is the distance between $\mathcal{A}'$ and $\mathcal{A}$ with $I = \pi/2$, $\omega = \pi/2$:

$$\delta_e(q, e) = \left[(\xi - q')^2 + \frac{q^2 e^2}{(1-e)^2}\right.$$

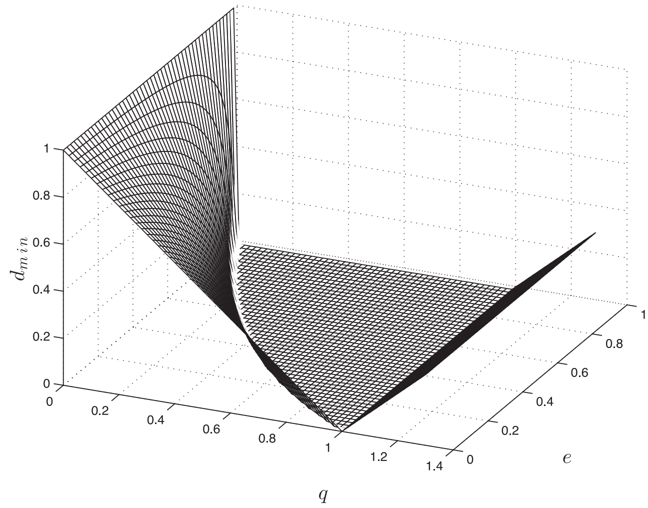

Figure 7. Graph of the function $(q, e) \mapsto \min_{\mathcal{D}_3} d_{\min}(q, e)$.

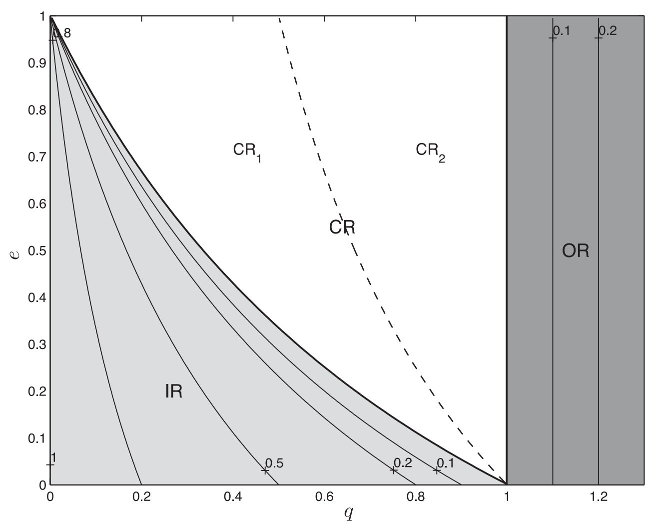

Figure 8. Partition of $\{(q, e) \in (0, q_{\max}] \times [0, 1]\}$ in the outer region (OR), crossing region (CR, composed by CR_1 and CR_2) and inner region (IR). We also plot the level curves of $(q, e) \mapsto \min_{\mathcal{D}_3} d_{\min}(q, e)$.

$$\left. -2\frac{qe}{(1-e)}\sqrt{\frac{q^2}{(1-e)^2} - \frac{\xi^2}{1-e^2}} + \frac{q^2}{(1-e)^2} - \frac{\xi^2}{1-e^2}\right],$$

where $\xi = \xi(q, e)$ is the unique real positive solution of

$$e^4 x^4 + 2q' e^2 (1-e^2) x^3 + (1+e)^2 (q'^2 (1-e)^2 + q^2 e^2) x^2$$
$$-2q' q^2 e^2 (1+e)^2 x - q'^2 q^2 (1-e^2)(1+e)^2 = 0.$$

See Section A3 for the details of this computation.

Proof. Lower bound: by Step 1 in Proposition 1, we can choose $I = 0$. Therefore, if $(q, e) \in \mathrm{IR}$ we have $d_{\min} = q' - Q$, if $(q, e) \in \mathrm{CR}$ we have $d_{\min} = 0$ and if $(q, e) \in \mathrm{OR}$ we have $d_{\min} = q - q'$.

Upper bound: by Step 1 in Proposition 1, we can choose $I = \pi/2$. By Step 2 in Proposition 1, we have $y'(f'_{\min}) = 0$. If $e = 0$, for every value of ω we get $d_{\min} = |q - q'|$. If $e \neq 0$, by the same Step 2, we must choose either $\omega = 0$ or $\omega = \pi/2$. For $\omega = 0$ we have $d_{\min} = \min\{|Q - q'|, |q' - q|\}$, and for $\omega = \pi/2$ we have $d_{\min} = \delta_e(q, e)$.

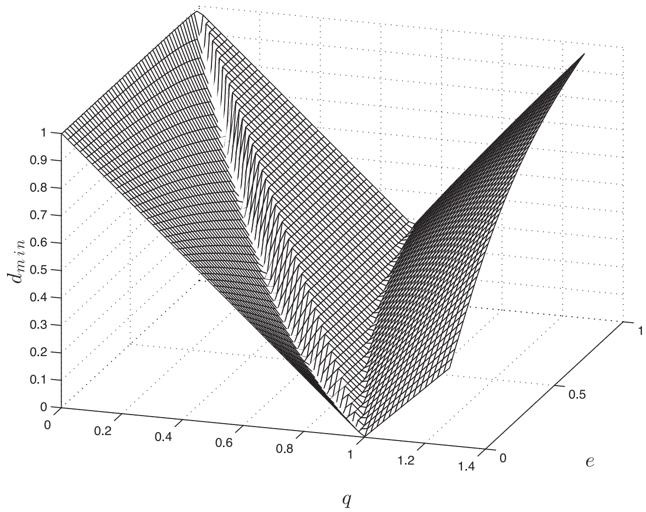

Figure 9. Graph of the function $(q, e) \mapsto \max_{\mathcal{D}_3} d_{\min}(q, e)$.

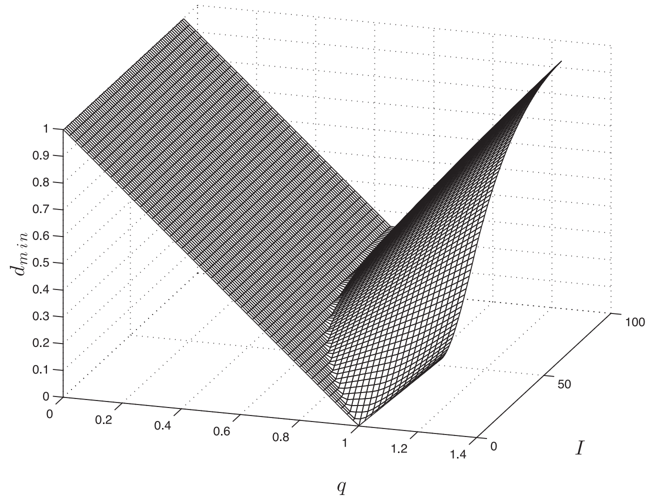

Figure 11. Graph of the function $(q, I) \mapsto \max_{\mathcal{D}_5} d_{\min}(q, I)$.

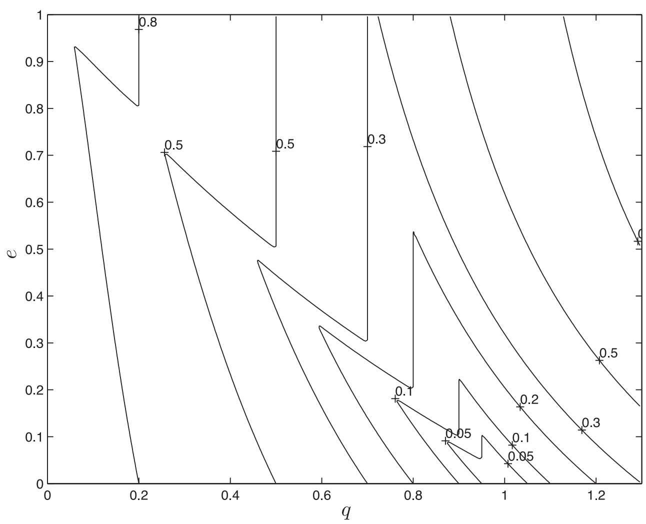

Figure 10. Level curves of $(q, e) \mapsto \max_{\mathcal{D}_3} d_{\min}(q, e)$.

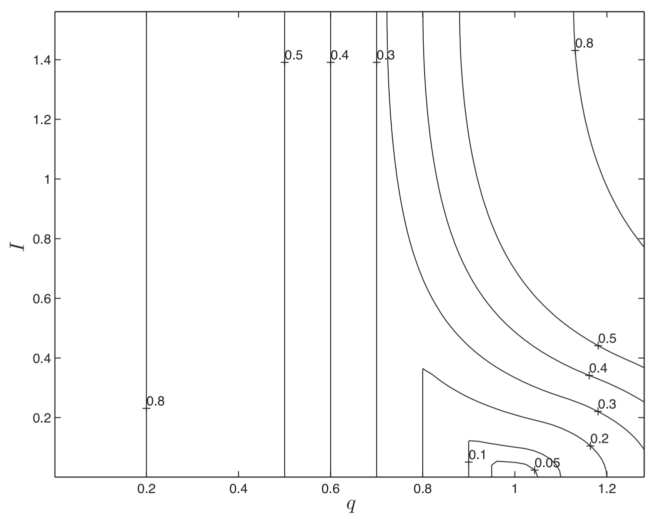

Figure 12. Level curves of $(q, I) \mapsto \max_{\mathcal{D}_5} d_{\min}(q, I)$.

We summarize the result with the following formula, which holds also for $e = 0$:

$$\max_{(I,\omega)\in\mathcal{D}_3} d_{\min} = \max\{\min\{|Q - q'|, |q' - q|\}, \delta_e(q, e)\}. \tag{20}$$

We observe that for $(q, e) \in$ IR we have $\delta_e(q, e) \geq q' - Q$ and for $(q, e) \in$ OR we have $\delta_e(q, e) \geq q - q'$. If $(q, e) \in$ CR then $q \leq q'$ and $Q \geq q'$; hence, (20) actually coincides with the second relation in (19). In Figs 9 and 10 we plot the graph and some level curves of the function $(q, e) \mapsto \max_{\mathcal{D}_3} d_{\min}(q, e)$.

Now we describe the possible values of $d_{\min}$ as a function of (q, I).

Proposition 4. Let $\mathcal{D}_5 = \{(e, \omega) : 0 \leq e \leq 1, 0 \leq \omega \leq \frac{\pi}{2}\}$, $\mathcal{D}_6 = \{(q, I) : 0 < q \leq q_{\max}, 0 \leq I \leq \frac{\pi}{2}\}$. For each choice of $(q, I) \in \mathcal{D}_6$ we have

$$\begin{cases} \min\limits_{(e,\omega)\in\mathcal{D}_5} d_{\min} = \max\{0, q - q'\} \\ \max\limits_{(e,\omega)\in\mathcal{D}_5} d_{\min} = \max\{q' - q, \delta_I(q, I)\} \end{cases}, \tag{21}$$

where $\delta_I(q, I)$ is the distance between $\mathcal{A}'$ and $\mathcal{A}$ with the constraints $e = 1$, $\omega = \pi/2$ (see Section A4).

Proof. Lower bound: if $q > q'$ then $\mathcal{A}$ is external to the ball $\mathcal{B}(0, q')$ with centre O and radius q', so that $d_{\min} \geq q - q'$. If we select $e = 0$, we obtain just $d_{\min} = q - q'$.

If $q \leq q'$, for each I we can find values of (e, ω) such that $d_{\min} = 0$.

Upper bound: if $I = 0$ the maximal orbit distance is $|q' - q|$ and corresponds to the second relation in (21) since, for $q > q'$, $\delta_I(q, 0) = q - q'$.

Assume now $I \neq 0$. If $q < q'/2$ then there is at least a node of $\mathcal{A}$ which is internal to $\mathcal{A}'$; in this case the maximal value of $d_{\min}$ is $q' - q$ and is attained for $e = 0$. In fact for each (e, ω) we have $d_{\min} \leq \min |d^{\pm}_{\rm nod}|$ and, in this case, $\min |d^{\pm}_{\rm nod}| \leq q' - q$.

If $q \geq q'/2$, we consider two subsets of $\mathcal{D}_5$: the first, denoted by $\mathcal{D}_{5,1}$, is the set of values of (e, ω) such that at least one node of $\mathcal{A}$ is internal to $\mathcal{A}'$ (white and bright-shadowed regions in Fig. 13, where $q = 0.7$ au). The second subset, denoted by $\mathcal{D}_{5,2}$, is the set of values of (e, ω) such that all the nodes of $\mathcal{A}$ are external (dark-shadowed region). In the boundary between $\mathcal{D}_{5,1}$ and $\mathcal{D}_{5,2}$, defined by the node

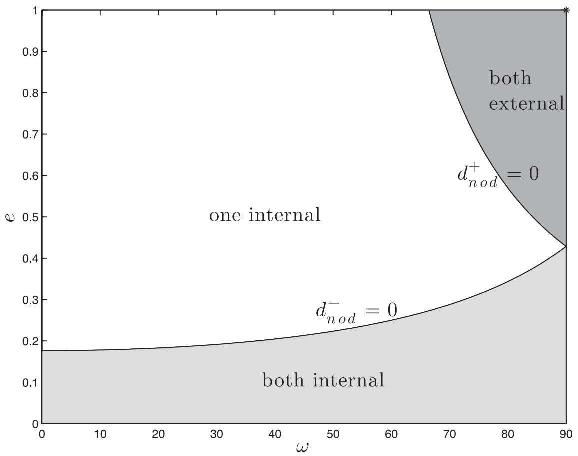

Figure 13. Node crossing curves and partition of the domain $\mathcal{D}_5$ for $q = 0.7$ au. The asterisk $\star$ (upper-right corner) indicates the point attaining the maximal value of $d_{\min}$ in the region with both nodes external. The maximal value of $d_{\min}$ in the region with at least one internal node is attained for $e = 0$.

crossing curves $d^{\pm}_{\rm nod} = 0$, we have $d_{\min} = 0$. Note that, if $q > q'$ the nodes are always external for every choice of (e, ω).

If we restrict $d_{\min}$ to $\mathcal{D}_{5,1}$ the proof works as in the case $q < q'/2$, and we obtain

$$\max_{\mathcal{D}_{5,1}} d_{\min} = q' - q \, ,$$

which is attained for $e = 0$.

Now we prove that the maximal value of $d_{\min}$ restricted to $\mathcal{D}_{5,2}$ is given by the orbit distance of a parabolic orbit ($e = 1$), with $\omega = \pi/2$.

In the proof, we shall use the following geometric property of an absolute minimum point of d^2. Let P be a point on $\mathcal{A}$ attaining the minimal value $d_{\min}$. Then, we can consider a circular torus $\mathbb{T}$ (see Fig. 14), with parametric equations

$$x = (q' + d_{\min}\cos\theta)\cos\phi \, ,$$

$$y = (q' + d_{\min}\cos\theta)\sin\phi \, ,$$

$$z = d_{\min}\sin\theta \, ,$$

with $\theta, \phi \in S^1$. The point P must lie on the torus $\mathbb{T}$ and the remaining part of the trajectory $\mathcal{A}$ must lie outside $\mathbb{T}$ (because the nodes are external), being allowed to touch tangentially the torus at some other point.

Let Q be the corresponding point on $\mathcal{A}'$ attaining the minimal distance, i.e. $|P - Q| = d_{\min}$. By the property of the stationary

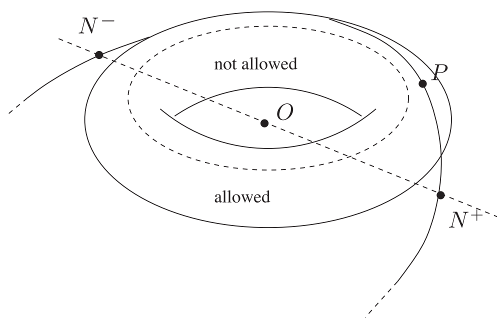

Figure 14. Geometric construction for the allowed positions of P on the torus $\mathbb{T}$, assuming that all the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$.

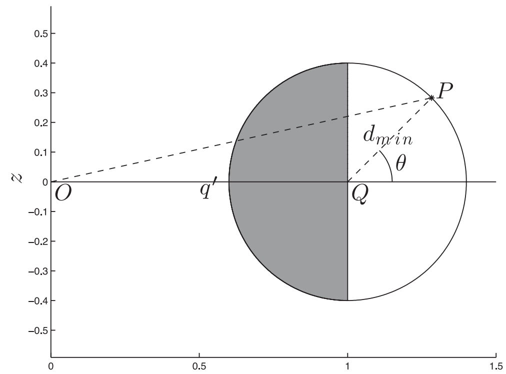

Figure 15. Section of the torus $\mathbb{T}$ containing the minimum points P, Q. The region not allowed for P is shaded.

points of d^2, since $\mathcal{A}'$ is circular, P must lie on a circle with centre Q and radius $d_{\min}$, which is obtained as a section of $\mathbb{T}$, orthogonal to $\mathcal{A}'$ (see Fig. 15). Moreover, since the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$, the θ coordinate of P is subject to the constraint

$$\cos\theta \geq 0. \qquad (22)$$

In fact, since in $\mathcal{D}_{5,2}$ the nodes of $\mathcal{A}$ are external, with $d^{\pm}_{\rm nod} \leq -d_{\min}$, if P lies in the region labelled as *not allowed* in Fig. 14, in which $\cos\theta < 0$, then there would be another point $\tilde{P}$ of $\mathcal{A}$ passing inside $\mathbb{T}$, so that the value of the minimal distance would be strictly less than $d_{\min}$.

First, we prove that necessarily $e = 1$. Choose $e < 1$ and let P, Q be two points on the two trajectories $\mathcal{A}$, $\mathcal{A}'$ attaining the minimal distance. From (14) we have

$$\frac{\partial \boldsymbol{r}}{\partial e} = \frac{\partial r}{\partial e}\hat{\boldsymbol{r}} \, ,$$

with $\frac{\partial r}{\partial e} \geq 0$, so that, using the continuous dependence on e of the stationary points of d^2, the orbit distance increases by increasing e. In fact, by slightly increasing e, all the points on the trajectory $\mathcal{A}$, in a neighbourhood of P, will increase their distance from $\mathcal{A}'$. The last statement is true because of relation (22). Note that this argument works also in the case of more than one absolute minimum point.

We conclude that, if $d_h = d_{\min}$ and $d^{\pm}_{\rm nod} \leq -d_{\min}$, then $\frac{\partial}{\partial e} d_h^2 \geq 0$; thus, the maximal value of $d_{\min}$ is attained for $e = 1$.

We now show that the maximal value of $d_{\min}$ in $\mathcal{D}_{5,2}$ is attained for $\omega = \pi/2$.

First, we consider the case $I = \pi/2$. Since we are assuming that the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$, then the statements in Steps 1 and 3 of Proposition 1 are fulfilled, so that, by Steps 2 and 4 of the same proposition, we have $\omega = \pi/2$.

If $I \neq \pi/2$, we can write

$$\frac{\partial d_h^2}{\partial \omega} = 4\left(\frac{y(f_h)}{\cos I} x'(f_h') - x(f_h) y'(f_h') \cos I\right)$$

$$= 4 x'(f_h') y(f_h) \frac{\sin^2 I}{\cos I} \, ,$$

where we have used $x(f_h) y'(f_h') = x'(f_h') y(f_h)$, which follows from the second equation in (5). We show that, if $d_h = d_{\min}$, we have $x'(f_h') y(f_h) \geq 0$, so that we can choose $\omega = \pi/2$.

By contradiction, assume $x'(f_h') y(f_h) < 0$. Then we have

$$(i) \begin{cases} x'(f_h') > 0 \\ y(f_h) < 0 \end{cases} \quad \text{or} \quad (ii) \begin{cases} x'(f_h') < 0 \\ y(f_h) > 0 \end{cases} . \qquad (23)$$

Table 1. Transformations producing a lower value of d.

	Range of $f_h+\omega$	Range of f'_h	Transformation
(i)	$(-\frac{\pi}{2},0)$	$(0,\frac{\pi}{2})$	$f'_h\to -f'_h$
	$(\pi,\frac{3}{2}\pi)$	$(-\frac{\pi}{2},0)$	$f'_h\to \pi-f'_h$
	$(\pi,\frac{3}{2}\pi)$	$(0,\frac{\pi}{2})$	$f'_h\to -f'_h$
(ii)	$(0,\frac{\pi}{2})$	$(\frac{\pi}{2},\pi)$	$f'_h\to \pi-f'_h$
	$(\frac{\pi}{2},\pi)$	$(\pi,\frac{3}{2}\pi)$	$f'_h\to -f'_h$
	$(0,\frac{\pi}{2})$	$(\pi,\frac{3}{2}\pi)$	$f'_h\to \pi-f'_h$

If (*i*) holds, then we can assume $f_h+\omega\in(-\pi/2,0)$, $f'_h\in(-\pi/2,0)$; in fact in the other cases we can select another value of f', keeping $f=f_h$, and obtain a value of d smaller than d_h. In a similar way, if (*ii*) holds, then we can assume $f_h+\omega\in(\pi/2,\pi)$, $f'_h\in(\pi/2,\pi)$.

In Table 1, we show a list of possible choices of f', depending on the values of (f_h,f'_h), that yield a value of d smaller than $d_h=d(f_h,f'_h)$. These results can be checked using the relation

$$d^2=q'^2+r^2-2q'r\cos\alpha\,,$$

where

$$\cos\alpha=\cos(f+\omega)\cos f'+\sin(f+\omega)\sin f'\cos I$$

and $\alpha=\alpha(f,f')$ is the angle between $\boldsymbol{r}(f)$ and the position vector $\boldsymbol{r}'(f')$ of a point on $\mathcal{A}'$.

Now we exclude the case

$$f_h+\omega\in(-\pi/2,0),\qquad f'_h\in(-\pi/2,0). \tag{24}$$

Such a pair (f_h,f'_h) cannot attain the minimal value $d_{\rm min}$ because in this case we would have [see Fig. 16(a)]

$$|d^+_{\rm nod}|=\frac{2q}{1+\cos\omega}-q'<\frac{2q}{1+\cos f_h}-q'$$
$$<\sqrt{\frac{4q^2}{(1+\cos f_h)^2}+q'^2-\frac{4qq'}{(1+\cos f_h)}\cos\alpha_h}=d_{\rm min}\,,$$

where $\alpha_h=\alpha(f_h,f'_h)$. Note that we have used the relation $q'<r(-\omega)<r(f_h)$, which is obtained from (24).

Finally, we exclude the case

$$f_h+\omega\in(\pi/2,\pi),\qquad f'_h\in(\pi/2,\pi). \tag{25}$$

We consider two cases: $f_h\le\omega$ and $f_h>\omega$. In the first case, we have $r(-\omega)\ge r(f_h)$. However, if we take $-f_h$ in place of f_h we have $r(f_h)=r(-f_h)$ and $0<z(-f_h)<z(f_h)$, so that the distance of the point corresponding to $\boldsymbol{r}(-f_h)$ from $\mathcal{A}'$ would be less than $d_{\rm min}$, which is a contradiction [see Fig. 16(b)]. In the second case, we find that $|d^+_{\rm nod}|$ would be smaller than $d_{\rm min}$. The geometric sketch of this last case is similar to the one in Fig. 16(a), here with $z(f_h)>0$; in fact also in this case we have $q'<r(-\omega)<r(f_h)$.

We observe that we have not used $e=1$ in the proof that the maximal orbit distance in $\mathcal{D}_{5,2}$ is attained for $\omega=\pi/2$. Therefore, this result also holds for $e<1$. In Figs 11 and 12 we plot the graph and some level curves of the function $(q,I)\mapsto\max_{\mathcal{D}_5}d_{\rm min}(q,I)$.

We conclude this section by describing the possible values of $d_{\rm min}$ as a function of q only.

Corollary 1. Let $\mathcal{D}_7=\{(e,I,\omega):e\in[0,1],\,I,\omega\in[0,\frac{\pi}{2}]\}$. For each $q\in(0,q_{\rm max}]$ we have

$$\begin{cases}\displaystyle\min_{(e,I,\omega)\in\mathcal{D}_7}d_{\rm min}=\max\{0,q-q'\}\\ \displaystyle\max_{(e,I,\omega)\in\mathcal{D}_7}d_{\rm min}=\max\{q'-q,\delta(q)\}\end{cases}, \tag{26}$$

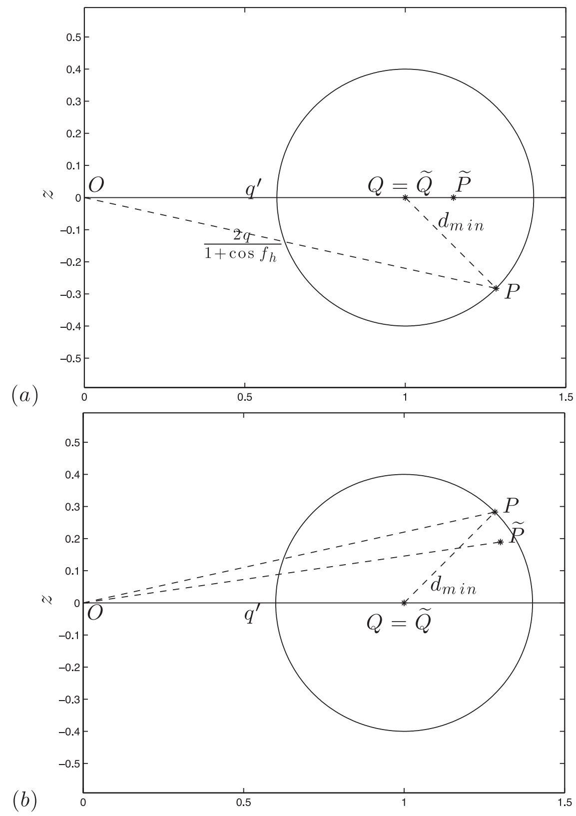

Figure 16. Geometry of the transformations decreasing the distance d. Here the two vertical sections of the torus $\mathbb{T}$ containing the relevant pairs of points (P,Q) and $(\tilde P,\tilde Q)$ are superposed. In (*a*) the pair $(\tilde P,\tilde Q)$ corresponds to the ascending node. In (*b*) the point $\tilde P$ is obtained from P by changing f_h into $-f_h$.

where $\delta(q)$ is the distance between $\mathcal{A}'$ and $\mathcal{A}$ with $e=1$, $I=\omega=\pi/2$,

$$\delta(q)=\delta_\omega(q,\pi/2)=\sqrt{(\xi-q')^2+\left(\frac{\xi^2-4q^2}{4q}\right)^2}\,,$$

with $\xi=\xi(q)$ the unique real solution of $x^3+4q^2x-8q'q^2=0$.

Proof. Lower bound: it follows immediately from Proposition 2.

Upper bound: from Step 1 in Proposition 1, we can choose $I=\pi/2$. From Step 2 in the same proposition the problem is reduced to the computation of the minimum of the distances between the nodes $Q_\pm$ of the Earth orbit and $\mathcal{A}$, and we obtain either $\omega=0$ or $\omega=\pi/2$. From Proposition 2, we know that we have either $e=0$ or $e=1$. Moreover, $e=1$ can yield the maximal orbit distance only in the case of external nodes, so that, by Step 3 in Proposition 1, we obtain that in this case we have $\omega=\pi/2$.

Thus, (26) holds, with

$$\delta(q)=\delta_\omega(q,\pi/2)\,,$$

where δ_ω is defined as in (16).

4 THE ORBITAL DISTRIBUTION OF THE KNOWN NEAs

We use the bounds introduced in Section 3.2 to explain some selection effects in the orbital distribution of the known NEAs.

In Fig. 17, we plot the pairs (q,ω) for the faint known NEAs, i.e. those with $H>22$. In the same figure, we draw the level curves

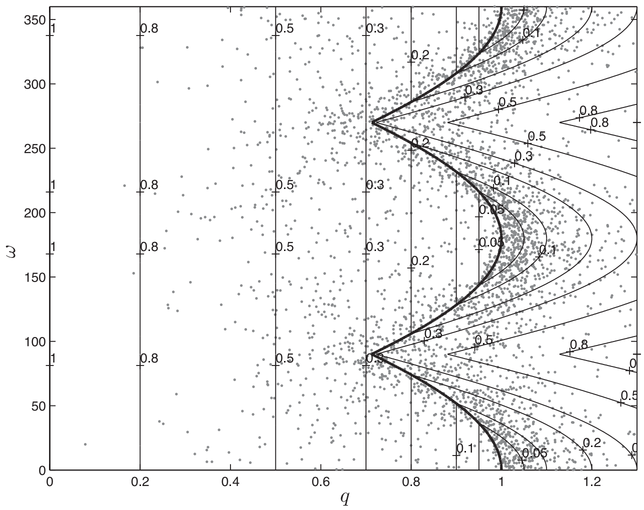

Figure 17. Distribution of the known NEAs with $H > 22$ in the plane (q, ω).

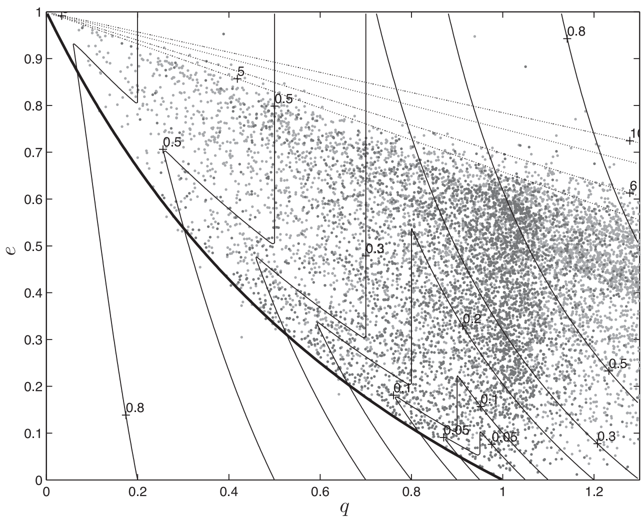

Figure 18. Distribution of all the known NEAs in the plane (q, e). The NEAs with $H > 22$ are plotted with darker grey.

of $(q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\min}(q, \omega)$, and the curve γ defined in (18). To plot Fig. 17, we have used the invariance of $\max_{\mathcal{D}_1} d_{\min}$ under the symmetries

$$\omega \to \pi - \omega\,, \quad \omega \to \pi + \omega\,, \quad \omega \to 2\pi - \omega\,,$$

for $\omega \in [0, \pi/2]$.

We observe that these asteroids are concentrated close to the curve γ (enhanced in Fig. 17), representing values of (q, ω) such that $\max_{\mathcal{D}_1} d_{\min}(q, \omega)$ is small (see Fig. 5). A reasonable explanation for that is the following: due to the geometric constraints, the faint asteroids with (q, ω) close to the curve γ are easier to detect, whatever their values of (e, I); in contrast, the ones with (q, ω) far from γ are not always detected. Moreover, the distribution of the values of ω for these asteroids can be assumed uniform. Therefore, the projection on to the plane (q, ω) yields this concentration effect.

In Fig. 18, we draw the distribution of all the known NEAs in the plane (q, e), together with the level curves of $(q, e) \mapsto \max_{\mathcal{D}_3}(q, e)$. The asteroids with $H > 22$ are plotted with darker grey. We can observe a concentration of these faint NEAs around $q = q'$. However, this case is different from the previous plot, since the distribution of

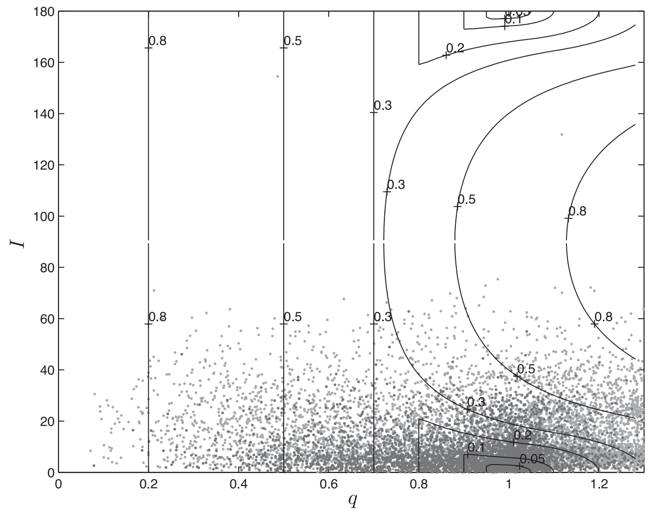

Figure 19. Distribution of the known NEAs in the plane (q, I). The NEAs with $H > 22$ are plotted with darker grey.

the values of e cannot be assumed uniform. The enhanced curve in Fig. 18 describes the boundary with the inner-Earth asteroids, i.e. the curve $q(1 + e)/(1 - e) = q'$, so that we have only a few NEAs below it. On the other hand, from Figs 7 and 8 we can see that the value of the minimal distance $\min_{\mathcal{D}_3} d_{\min}$ steeply increases as we go below this curve.

The lack of asteroids in the upper part of Fig. 18 cannot be explained in terms of orbit distance. The straight lines passing through $(q, e) = (0, 1)$ correspond to different periods (labelled in years) for the asteroid orbit. Thus, waiting for future survey operations, some asteroids which have not been discovered yet should appear in this portion of the plane (q, e). This time limitation is unavoidable.

In Fig. 19, we draw the distribution of all the known NEAs in the plane (q, I), together with the level curves of $(q, I) \mapsto \max_{\mathcal{D}_5}(q, I)$. The asteroids with $H > 22$ are plotted with darker grey. To plot this figure, we have used the invariance of $\max_{\mathcal{D}_5} d_{\min}$ under the symmetry $I \to \pi - I$ for $I \in [0, \pi/2]$. Here, we can only observe a concentration of very faint NEAs around $q = q' = 1$ au, with low inclinations. However, we recall that the distribution of the values of I is very far from uniform.

In Fig. 20, we draw the distribution of the known NEAs in the plane $(q, d_{\min})$. The maximal value of $d_{\min}$ found for the 9220 known NEA orbits (to the date of 2012 October 21) is 0.7036 au, attained by asteroid 2010 KY_{127}.

The V-shaped structure appearing in Fig. 20 corresponds to the graph of $q \to |q - q'|$, and can be understood by looking at the projection in the direction of ω of the minimal and maximal orbit distance surfaces $\min_{\mathcal{D}_1} d_{\min}$, $\max_{\mathcal{D}_1} d_{\min}$ defined in Proposition 2 (see Fig. 5).

5 THE THRESHOLD OF THE NEA CLASS

In Fig. 21, we show how the maximal value of $d_{\min}$ varies by a small change of the threshold value of $q_{\max}$, defining the NEA class. Note that the value $q_{\max} = 1.3$ au, introduced in Shoemaker & Helin (1978), is very peculiar. Actually if we choose e.g. $q_{\max} = 1.299$ au, the maximal value of $d_{\min}$ becomes 1 au; this value is obtained by a Sun-grazing trajectory, perpendicular to $\mathcal{A}'$.

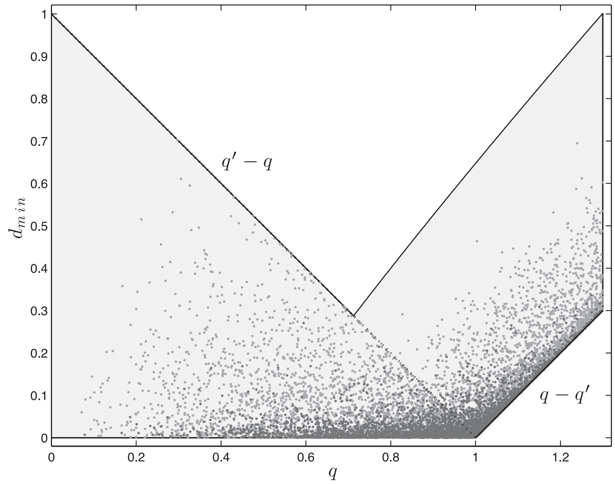

Figure 20. Distribution of the known NEAs in the plane (q, $d_{\min}$). The shaded region represents the possible values of $d_{\min}$ as a function of q. The NEAs with $H > 22$ are plotted with darker grey.

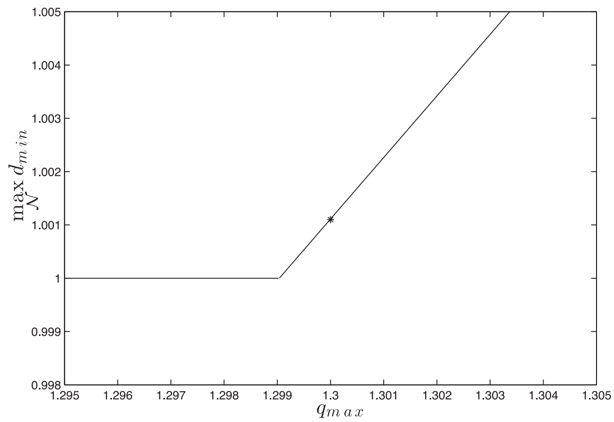

Figure 21. The maximal orbit distance as a function of $q_{\max}$.

6 APPARENT MAGNITUDE AT THE MAXIMAL ORBIT DISTANCE

Every survey has a limiting apparent magnitude: this defines a threshold for the size of the NEAs that can be discovered. In fact, the apparent magnitude H_{app} is related to the absolute magnitude H_{abs}, and through the latter we obtain an estimate of the asteroid diameter. We use an approximated formula for this relation, taken from Bowell et al. (1989), that we recall below. Given the heliocentric and geocentric position of the asteroid ($\mathcal{X}$, $\mathcal{X} - \mathcal{X}'$, respectively), we can compute the phase angle β (i.e. the angle Sun–Asteroid–Earth). If we fix a value for the slope parameter G (depending on the *albedo* of the asteroid surface), we have

$$H_{\mathrm{app}} = H_{\mathrm{abs}} + 5\log_{10}(r\rho) - 2.5\log_{10}((1-G)\phi_1 + G\phi_2)\,,$$

where

$$r = |\mathcal{X}|\,, \quad \rho = |\mathcal{X} - \mathcal{X}'|\,,$$

$$\phi_j = \exp\left[-a_j\left(\tan\frac{\beta}{2}\right)^{b_j}\right], \quad j = 1, 2\,,$$

$$a_1 = 3.33\,,\ a_2 = 1.87\,,\ b_1 = 0.63\,,\ b_2 = 1.22.$$

We consider an object on the parabolic trajectory $\mathcal{P}$, defined in Step 3 of Proposition 1. If we set $G = 0.15$, which corresponds to a typical value for asteroids, and assume that this object is observed from the Earth when the two bodies are at the minimal distance points, then we have

$$\mathcal{X} \approx (1.5, 0, 0.867)\ \mathrm{au}\,, \quad \mathcal{X}' \approx (1, 0, 0)\ \mathrm{au}\,,$$

so that

$$r \approx 1.73\ \mathrm{au}\,, \qquad \rho \approx 1.0011\ \mathrm{au}.$$

In conclusion, we obtain

$$\beta \approx 30^\circ\,, \qquad H_{\mathrm{app}} \approx H_{\mathrm{abs}} + 2.495.$$

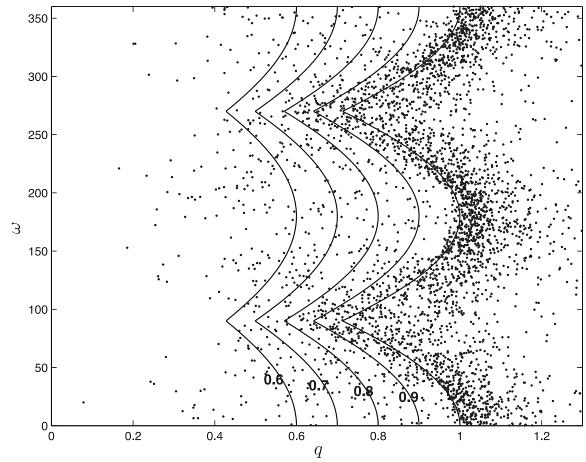

Figure 22. Comparison among the γ curves corresponding to $q' = 0.6, 0.7, 0.8, 0.9, 1$ au. We also plot the known NEAs with $H > 22$.

7 CONCLUSIONS

We have proven certain geometric properties of confocal conics and shown that they can explain some selection effects in the orbital distribution of the known NEAs. These asteroids have been detected for the large majority with ground-based observations. The results of this paper are relevant to plan future surveys, with space-based observations; in fact, the optimal bounds in Propositions 2, 3, 4 hold for arbitrary values of $q' > 0$. Thus, for example, placing a space telescope in a circular orbit with radius equal to the average semimajor axis of Venus ($q' \approx 0.72$ au), we should discover several faint NEAs. The suggestion of placing telescopes in satellite orbits much interior to the Earth, to increase the NEA discovery, can also be found in Jedicke et al. (2003) and Martinot & Morbidelli (2006). In Fig. 22, we show the γ curves corresponding to different choices of q', with the background of the known fainter NEAs ($H > 22$).

Finally, we note that the surfaces of minimal and maximal distance can be useful in a stress test for a program computing the orbit distance. In fact one could perform a very large number of orbit distance computations, with one circular orbit, and check if the computed values lie within the bounds of Propositions 2–4.

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APPENDIX A: COMPLEMENTARY RESULTS AND COMPUTATIONS

A1 A result by Apollonius

The distance between a conic section and a point in the plane was studied very long ago (≈200 BC) by Apollonius of Perga, see Heat (1981). Here we recall some of his results in modern mathematical language (see Hartmann & Jantzen 2004). Choose rectangular coordinates (x, y) in the plane and consider the ellipse $\mathcal{A}$, defined by the equation

$$\frac{[(1-e)x+eq]^2}{q^2}+\frac{y^2(1-e)}{q^2(1+e)}=1\,,$$

where q, e are the pericentre distance and the eccentricity of $\mathcal{A}$, respectively.

Moreover, let Q be the point with coordinates $(x_0, y_0) = (q'\cos\omega, q'\sin\omega)$. We want to compute the minimum value of the distance d_Q between Q and a point on $\mathcal{A}$, which can be parametrized by the true anomaly f, obtaining in this way a function $f\mapsto d_Q(f)$. We can consider all the normals from Q to the ellipse $\mathcal{A}$; the intersections of these normals with $\mathcal{A}$ correspond to the stationary points of $f\mapsto d_Q(f)$.

The hyperbola $\mathcal{H}$, defined by

$$[(1-e)x+eq][ye^2+y_0(1-e^2)]-[(1-e)x_0+eq]=0\,,$$

has the asymptotes parallel to the coordinate axes and one branch passing through Q and the centre O of the ellipse (see Fig. A1). Moreover, $\mathcal{H}$ intersect $\mathcal{A}$ just in the stationary points of d_Q. It turns out that the d_Q has at most four stationary points and at most two minima. The case with two minima occurs when both branches of $\mathcal{H}$ intersect $\mathcal{A}$ so that, due to the alternation of maxima and minima of $d_Q(f)$, there is one minimum point per branch. Apollonius also introduced the *evolute* of $\mathcal{A}$, which is a curve with the parametric equation

$$x(\phi)=\frac{qe^2}{1-e}\cos^3\phi\,,\qquad y(\phi)=-\frac{qe^2}{(1-e)\sqrt{1-e^2}}\sin^3\phi\,,$$

with $\phi\in S^1$. The evolute of the ellipse is a closed, star-shaped curve. Moreover, both branches of $\mathcal{H}$ intersect $\mathcal{A}$ if and only if Q lies inside the graph of the evolute (like in Fig. A1).

In Step 2 of Proposition 1, we study the distance between Q and $\mathcal{A}$ as the argument of perihelion ω varies in $[0, \pi]$. To this aim we

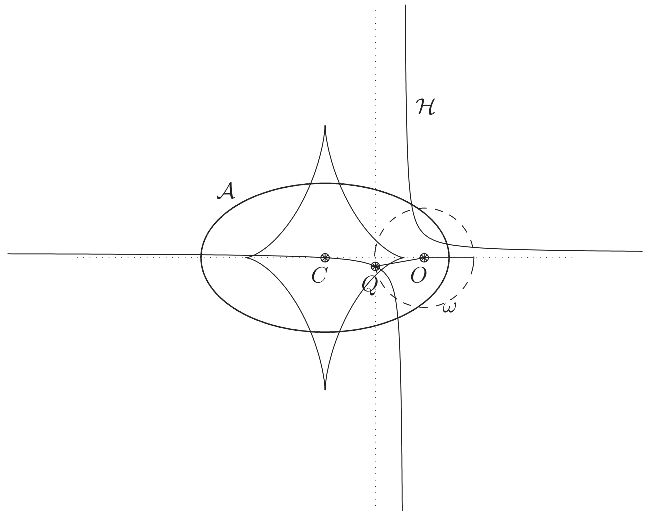

Figure A1. The two branches of Apollonius' hyperbola $\mathcal{H}$ in a case with four intersections with the ellipse $\mathcal{A}$. The star-shaped figure is the graph of the evolute of $\mathcal{A}$.

can of course leave $\mathcal{A}$ fixed and rotate by $-\omega$ the point Q around the focus O, see Fig. A1.

From these geometric results it is easy to deduce that an exchange of role between local minima as absolute minimum of $d_Q(f)$ can occur only for $\omega = 0, \pi$.

A similar statement also holds if $\mathcal{A}$ is a parabola. If the equation of $\mathcal{A}$ is

$$x-\frac{4q^2-y^2}{4q}=0\,,$$

then the normals from Q to $\mathcal{A}$ are at most three, and they are found by looking for the intersections between $\mathcal{A}$ and the hyperbola

$$xy-(2q+x_0)y+2qy_0=0.$$

A2 The curve γ

We describe the procedure to compute the curve γ in the plane (q, ω), given in equation (18).

System (17) can be written as $p_1 = p_2 = 0$, where

$$p_1(x)=x^4+8\,q(q'\cos\omega+q)x^2-32\,q^2q'\sin\omega\,x$$
$$\quad-\,32\,q^3q'(\cos\omega-1)$$

$$p_2(x)=x^3+4q(q'\cos\omega+q)x-8q^2q'\sin\omega$$

are polynomials in the x variable. By computing the resultant $Res(p_1, p_2, x)$ of p_1, p_2 with respect to x, the dependence on $\sin\omega$ disappears. Setting $y = \cos\omega$, we obtain

$$\begin{aligned}\frac{Res(p_1,p_2,x)}{-4096q^7q'(y-1)}&=2q^4+2q'(7\,y-5)q^3\\&\quad-2q'^2(3y+22)(y-1)q^2+q'^3\\&\quad\times(y^3+13y^2+9y-27)q-2y^3q'^4\\&=q'^3(-2q'+q)y^3-q'^2q(6q-13q')y^2\\&\quad+q'q(9q'^2-38q'q+14q^2)y\\&\quad+q(2q^3-27\,q'^3+44\,q'^2q-10q^2q').\end{aligned}$$

The factor $q^7(y-1)$ appears because δ_ω is equal to $|q-q'|$ for $y=1$ or $q=0$; however, this factor must be eliminated to obtain the algebraic expression for γ.

A3 Computation of $\boldsymbol{\delta_e(q, e)}$

We compute the orbit distance between $\mathcal{A}'$ and a trajectory $\mathcal{A}$ with $I=\pi/2$, $\omega=\pi/2$, as a function of (q, e).

From Step 2 of Proposition 1, we have to compute the distance of a nodal point $Q_\pm \in \mathcal{A}'$ from the trajectory $\mathcal{A}$. To this aim we can consider only the portion of $\mathcal{A}$ which is the graph of the function

$$x \mapsto z(x) = \frac{qe}{1-e} - \sqrt{\frac{q^2}{(1-e)^2} - \frac{x^2}{1-e^2}}\,,$$

where

$$x \in \left[-q\sqrt{\frac{1+e}{1-e}}, q\sqrt{\frac{1+e}{1-e}}\right].$$

We introduce

$$D^2(x) = (x-q')^2 + z^2(x)$$

and consider the stationarity condition

$$(D^2)' = 0. \tag{A1}$$

From (A1), we obtain the polynomial equation

$$\begin{aligned} &e^4x^4 + 2q'e^2(1-e^2)x^3 + (1+e)^2(q'^2(e-1)^2 + q^2e^2)x^2 \\ &-2q'e^2q^2(1+e)^2x - q'^2q^2(1-e^2)(1+e)^2 = 0. \end{aligned} \tag{A2}$$

By Descartes' rule of signs, equation (A2) has only one real positive solution $\bar{x}$, which corresponds to a component of an absolute minimum point.

A4 Computation of $\boldsymbol{\delta_I(q, I)}$

We compute the orbit distance between $\mathcal{A}'$ and a trajectory $\mathcal{A}$ with $e=1$, $\omega=\pi/2$, as a function of (q, I).

From the proof of Proposition 4, we know that we are interested only in the case with $I \neq 0$ and the nodes of $\mathcal{A}$ external to $\mathcal{A}'$. We also know that $y'_{\min} < 0$ (it follows from the proof that $\partial d_h^2/\partial\omega \geq 0$ if $d_h = d_{\min}$).

We use the following parametrization:

$$x' = -q'\sin\eta'\,, \quad y' = q'\cos\eta'\,, \qquad \eta' = f' - \pi/2\,;$$

$$y = \zeta\cos I\,, \quad z = \zeta\sin I\,, \qquad \zeta = q - \frac{x^2}{4q}.$$

Then we set

$$d^2(x, \eta') = (x-x')^2 + (y-y')^2 + z^2$$

and compute the stationary points of d^2:

$$4q^2\frac{\partial d^2}{\partial x} = 4q^2x + 8q^2q'\sin\eta' + 4qq'\cos I\,x\cos\eta' + x^3 = 0\,,$$

$$2\frac{q}{q'}\frac{\partial d^2}{\partial\eta'} = 4qx\cos\eta' - \cos I\,x^2\sin\eta' + 4q^2\cos I\sin\eta' = 0.$$

Apply the coordinate change

$$s = \tan(\eta'/2)\,, \tag{A3}$$

so that

$$\cos\eta' = \frac{1-s^2}{1+s^2}\,, \qquad \sin\eta' = \frac{2s}{1+s^2}.$$

Note that, since $\eta' = f' + \pi/2$, the point corresponding to $\eta' = \pi$, which is sent to infinity by (A3), cannot correspond to the f' component of the absolute minimum point, not even for $I=\pi/2$.

We obtain the polynomial equations

$$p_1(x, s) = 2qx(1-s^2) - s\cos I(x^2-4q^2) = 0\,,$$

$$\begin{aligned} p_2(x, s) = {} &(x^2+4q^2)x(1+s^2) + 16q^2q's \\ &+ 4qq'\cos I\,x(1-s^2) = 0. \end{aligned}$$

The resultant of p_1, p_2 with respect to s is the polynomial

$$\begin{aligned} \mathrm{res}(x) = {} &x^2[\cos^2 I\,x^8 + 16q^2x^6 \\ &- 16q^2(2q^2(\cos^2 I - 4) + q'^2\cos^4 I)x^4 \\ &+ 128q^4(2q^2 + q'^2\cos^2 I(\cos^2 I - 2))x^2 \\ &- 256q^6(q'^2(\cos^2 I - 2)^2 - q^2\cos^2 I]. \end{aligned}$$

For each root $\bar{x}$ of $\mathrm{res}(x)$ we search for the value $\bar{s}$ of s corresponding to a stationary point:

$$2qxs^2 + \cos I(x^2-4q^2)s - 2qx = 0. \tag{A4}$$

Equation (A4) has the roots

$$\bar{s}_{1,2} = \frac{-\cos I(x^2-4q^2) \pm \sqrt{\Delta}}{4qx}\,,$$

with

$$\Delta = \cos^2 I(x^4 - 8q^2x^2 + 16q^4) + 16q^2x^2.$$

To search for $d_{\min}$, the absolute minimum $d(x, \eta')$, we can restrict to

$$\eta' \in [-\pi/2, 0]\,, \qquad x \geq 0.$$

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